

# Dynamics of integrable birational maps preserving genus 0 foliations.

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## Introduction and some results

A planar *rational* map  $F : \mathcal{U} \rightarrow \mathcal{U}$ , where  $\mathcal{U} \subseteq \mathbb{K}^2$  is an open set and  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ , is called *birational* if it has a rational inverse  $F^{-1}$ . Also, we will say that a map  $F$  is *integrable* if there exists a non-constant function  $V : \mathcal{U} \rightarrow \mathbb{K}$  such that  $V(F(x, y)) = V(x, y)$  which is called a *first integral*. In this work, we will consider only integrable maps that have *rational first integrals*  $V(x, y) = V_1(x, y)/V_2(x, y)$ . In other words, the map preserves the foliation of  $\mathcal{U}$  given by the algebraic curves

$$C_h = \{V_1(x, y) - hV_2(x, y) = 0\}_{h \in \text{Im}(V)}.$$

It is known that any birational map  $F$  preserving a foliation of *nonsingular* curves of generic genus greater or equal than 2 is a globally periodic map, as a consequence of Hurwitz (1893) and Montgomery (1937) theorems. If the genus is generically 1, either  $F$  or  $F^2$  are conjugate to a linear action.

Our goal is to derive a methodology to study the genus 0 case.

The Cayley-Riemann Theorem ensures that genus 0 curves are rationally parametrizable. Furthermore, this parametrization can be chosen to be *proper*, i.e. a birational, which is unique modulus Möbius transformations, [2]. Hence we obtain:

### Proposition

Let  $F$  be a planar birational map preserving a foliation of algebraic curves  $\{C_h\}$  of genus 0, and  $\{P_h\}$  a family of proper parametrizations of  $\{C_h\}$ . Then, each  $F|_{C_h}$  is conjugated to a Möbius map  $M_h$ , where  $M_h = P_h^{-1} \circ F \circ P_h$ .

This is because each  $P_h$  is birational, so  $M_h$  is a one-dimensional birational map.

Since every rational real curve can be properly parametrized over the reals [2], then *real* birational maps preserving a foliation of real algebraic curves of genus 0 can be represented by *real* Möbius transformations.

### Corollary

Let  $F$  be a *real* planar birational map preserving a foliation of algebraic curves  $\{C_h\}$  of genus 0, and  $\{P_h\}$  a family of proper *real* parametrizations of  $\{C_h\}$ . Then each  $F|_{C_h}$  is conjugated to a Möbius map  $M_h$ , with real coefficients.

As the dynamics of the Möbius maps is well-known, we are able to give a description of the global dynamics of our maps, and in particular, the *explicit* expression of the *rotation number function* associated to given invariant curves in an open set foliated by closed ones. The explicitness of the rotation number function allow us to fully characterize the *set of periods* of these maps.

We also have proved that our maps have an associated *Lie symmetry*. Specifically,

### Theorem

Any birational map  $F$  preserving a foliation given by algebraic curves of genus 0,  $\{C_h\}$  where  $h = V(x, y)$ , has a Lie symmetry. Furthermore, if  $\{P_h(t) = (P_{1,h}(t), P_{2,h}(t))\}$  is a family of proper parameterizations of  $\{C_h\}$ , then there is a *Lie symmetry* of  $F$  given by the vector field in  $\mathbb{K}^2$

$$X(p) = P'_{1,h}(t)(P_h^{-1}(p))Y(P_h^{-1}(p))\frac{\partial}{\partial x} + P'_{2,h}(t)(P_h^{-1}(p))Y(P_h^{-1}(p))\frac{\partial}{\partial y}$$

where  $p = (x, y)$ , and  $Y(t) = -b(h) + (d(h) - a(h))t + c(h)t^2$ , is the Lie symmetry associated to the Möbius map

$$M_h(t) = \frac{a(h)t + b(h)}{c(h)t + d(h)},$$

given by  $M_h(t) = P_h^{-1} \circ F \circ P_h(t)$ .

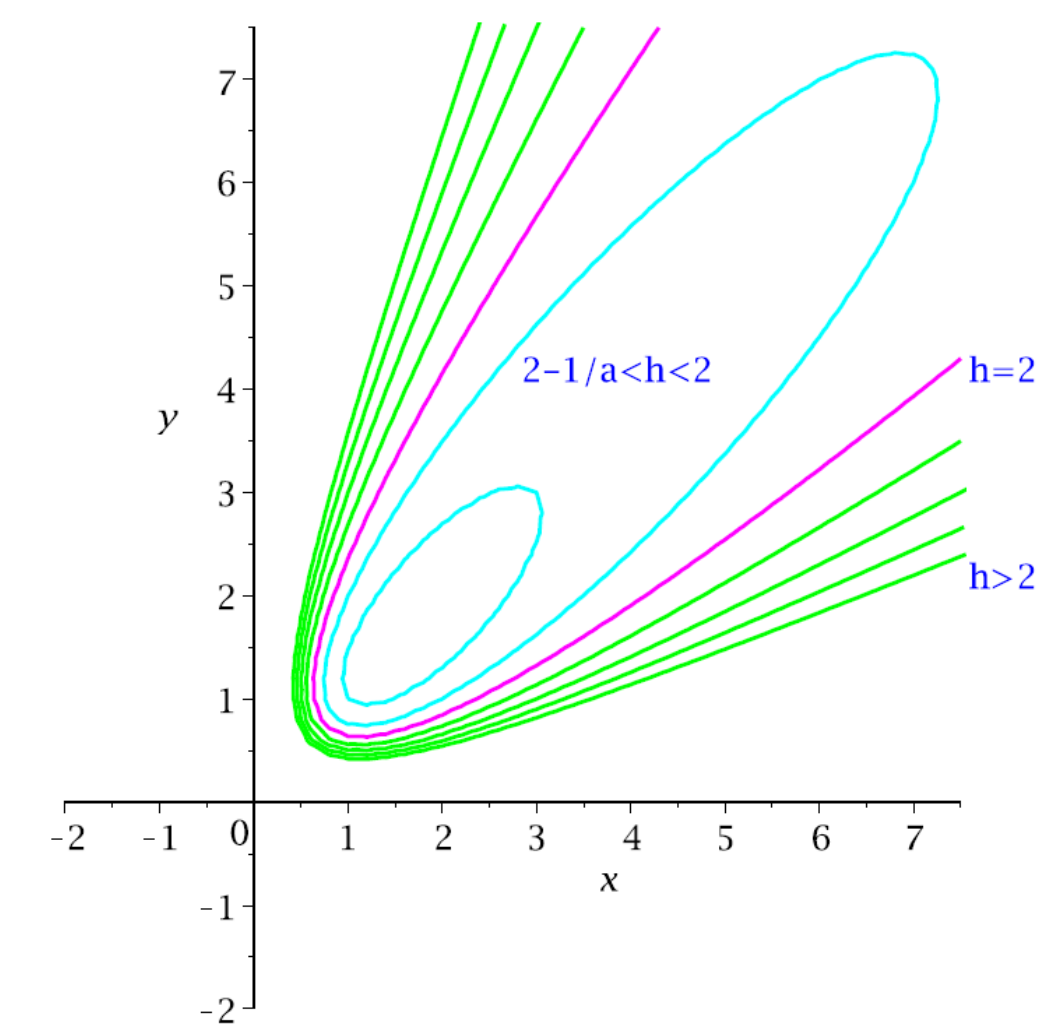
## Example: The Bastien and Rogalski map

We consider the planar birational map defined in  $\mathbb{R}^{2,+}$ .

$$F_a(x, y) = \left(y, \frac{a - y + y^2}{x}\right) \text{ with } a > 1/4,$$

This map was studied in [1]. It preserves the foliation given by the algebraic curves of genus 0:

$$C_h = \{x^2 + y^2 - x - y + a - hxy = 0\}.$$



Using an appropriate algorithm we find the *proper real* parametrization of  $C_h$  given by  $P_h(t) = (P_{1h}(t), P_{2h}(t))$  where

$$P_{1h}(t) = \frac{2\delta t - ah^2 - h\delta - h - 2 + 4a}{2(t^2 + ht - 1)} + a,$$

$$P_{2h}(t) = \frac{(-ah + \delta - 1)t^2 + (4a - 2)t - ah - \delta - 1}{2(t^2 + ht - 1)},$$

and  $\delta = \sqrt{(ha + 1 - 2a)(ha + 1 + 2a)}$ . Its inverse is

$$P_h^{-1}(x, y) = \frac{-2\delta x + (ah^2 + h\delta - 4a + h + 2)y - ah + 2a + \delta - 1}{(ah^2 - h\delta - 4a + h + 2)x + 2\delta y + ah - 2a - \delta + 1}.$$

Then, as show us the previous results, the composition  $M_h = P_h^{-1} \circ F \circ P_h$  gives

$$M_h(t) = \frac{(h+1)t - 1}{t + 1},$$

a Möbius map with real coefficients, and the Lie symmetry

$$X(x, y) = \frac{x^2 - y^2 + a - x}{y} \frac{\partial}{\partial x} + \frac{x^2 - y^2 - a + y}{x} \frac{\partial}{\partial y}.$$

Both of these results allow us to give a global description of the dynamics:

### Proposition

Set  $a > 1/4$ . Let  $\tilde{C}_h$  and  $\tilde{F}$  denote the extensions of  $C_h$  and  $F$  to  $\mathbb{R}P^2$ . Then, we get:

- For  $h > 2$ ,  $C_h$  is a hyperbola,  $F|_{C_h}$  is conjugated to a *translation*, and there are two fixed points of  $\tilde{F}|_{\tilde{C}_h}$  at infinity, an attractor and a repeller.
- For  $h = 2$ ,  $C_h$  is a parabola,  $F|_{C_h}$  is conjugated to a *translation*, and there is a non stable attractor of  $\tilde{F}|_{\tilde{C}_h}$  at infinity.
- For  $2 - 1/a < h < 2$ ,  $C_h$  is an ellipse,  $F|_{C_h}$  is conjugated to a *rotation* with explicit rotation number

$$\theta(h) = \arg \left( \frac{h - i\sqrt{4 - h^2}}{2} \right) \mod 2\pi.$$

Using this rotation number we reobtain the results in [1]:

### Proposition

Set  $a > 1/4$ ,  $h_c = 2 - 1/a$ , and  $\rho_a = \frac{1}{2\pi} \arg \left( \frac{2a-1-i\sqrt{4a-1}}{2a} \right)$ . Then, we have:

- For any fixed  $a > 1/4$  and  $p \geq E(1/(1 - \rho_a)) + 1$  there exists  $h_p \in (h_c, 2)$  such that  $C_{h_p}$  is filled of  $p$ -periodic orbits.
- For all  $p \in \mathbb{N}$ ,  $p \geq 3$  there exists  $a > 1/4$  and  $h_p \in (h_c, 2)$  such that  $C_{h_p}$  is filled of  $p$ -periodic orbits.

## References

- [1] G. Bastien, M. Rogalski. *On some algebraic difference equations  $u_{n+2}u_n = \psi(u_{n+1})$  in  $\mathbb{R}^+$ , related to families of conics or cubics: generalization of the Lyness' sequences*, J. Math. Anal. Appl. 300 (2004), 303–333.
- [2] J.R. Sendra, F. Winkler, S. Pérez-Díaz. Rational Algebraic Curves. Springer, New York 2008.